Constructive Galois Connections

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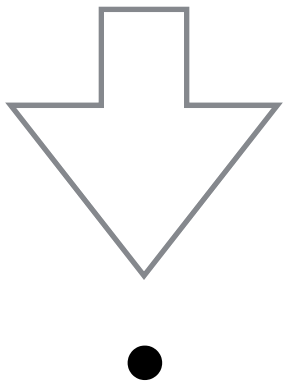
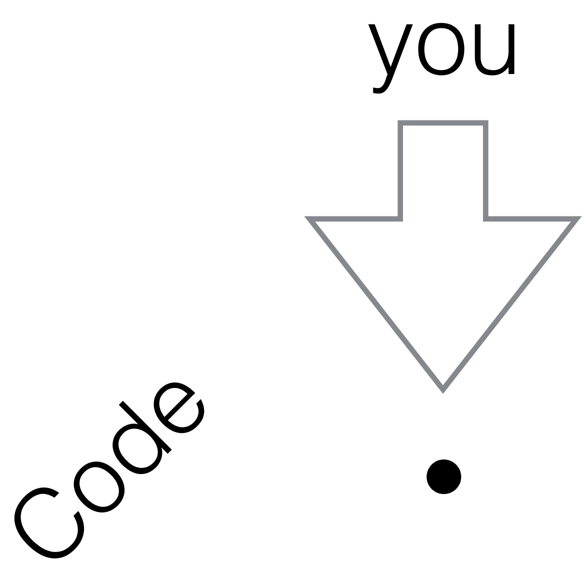
you

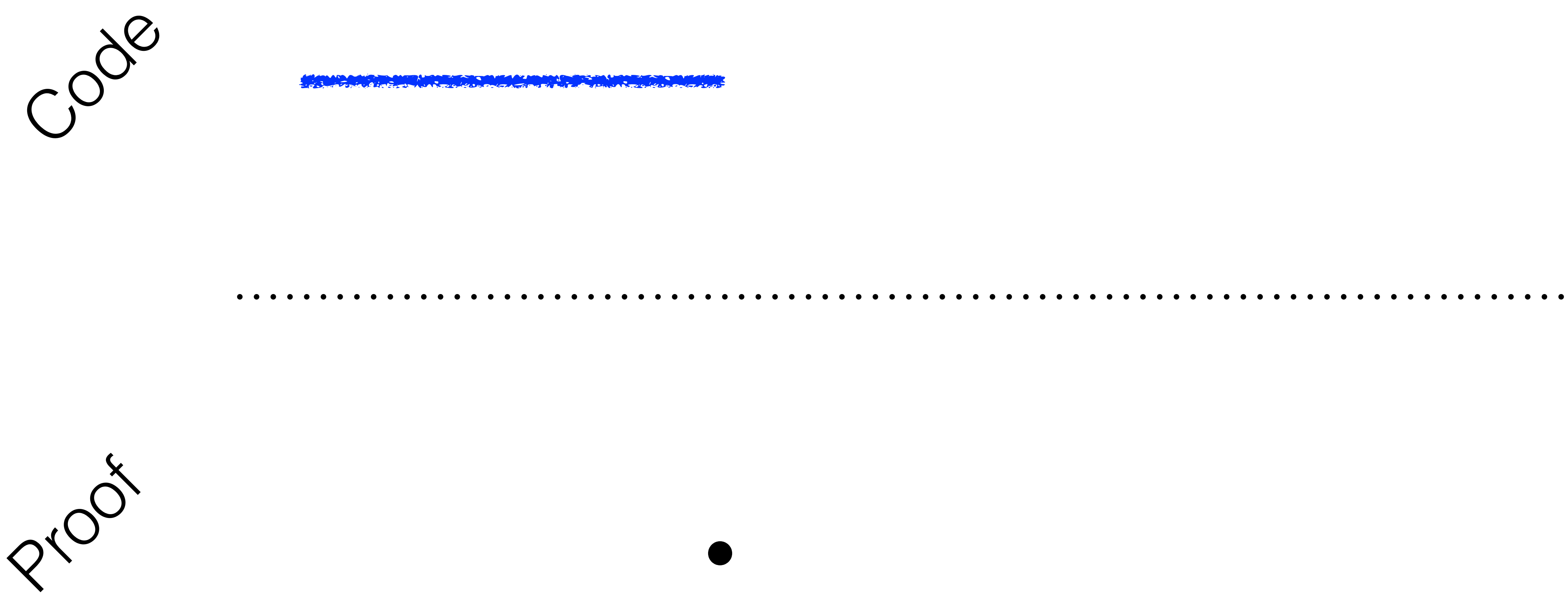



you
Code


Code

Code

Proof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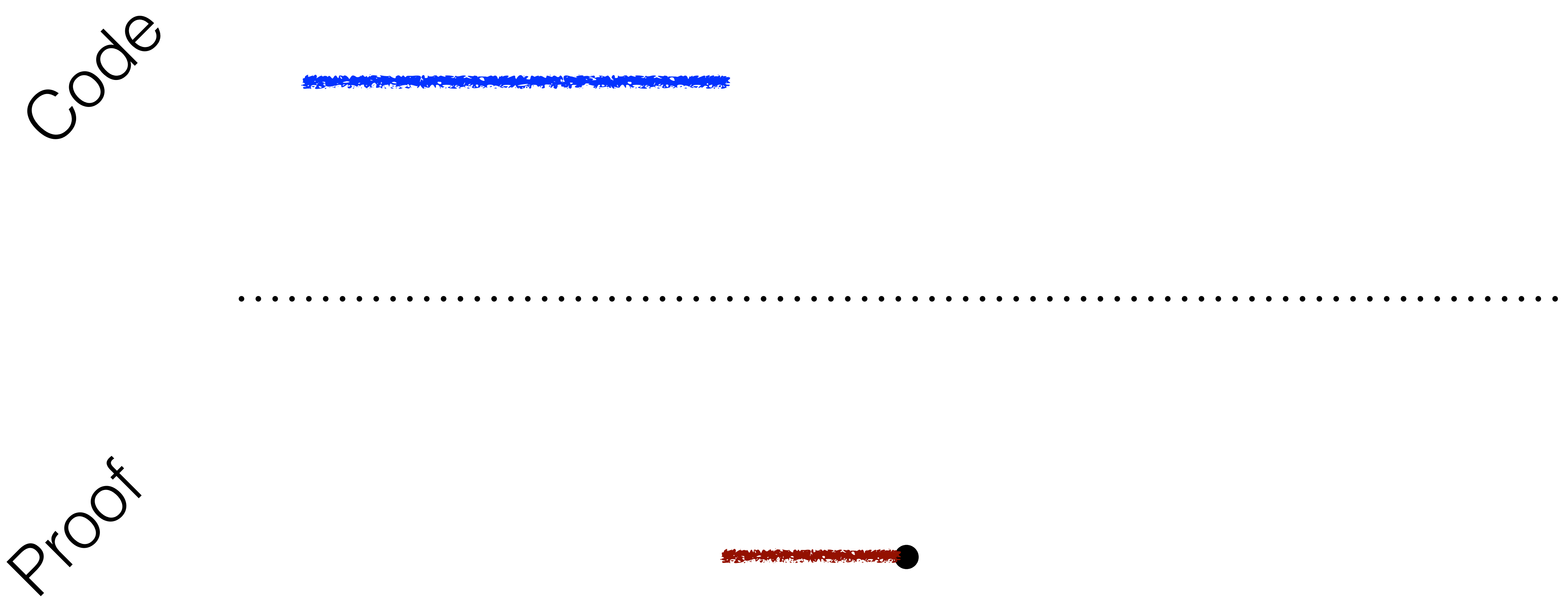
Proof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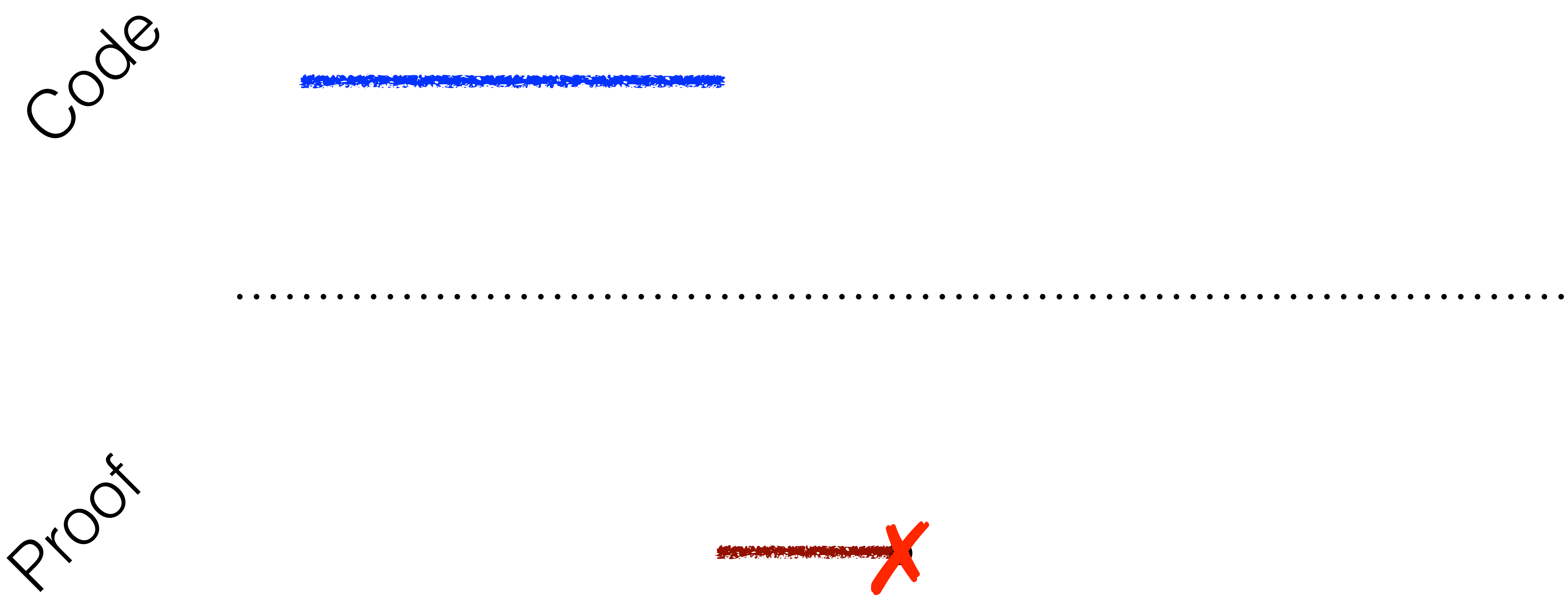
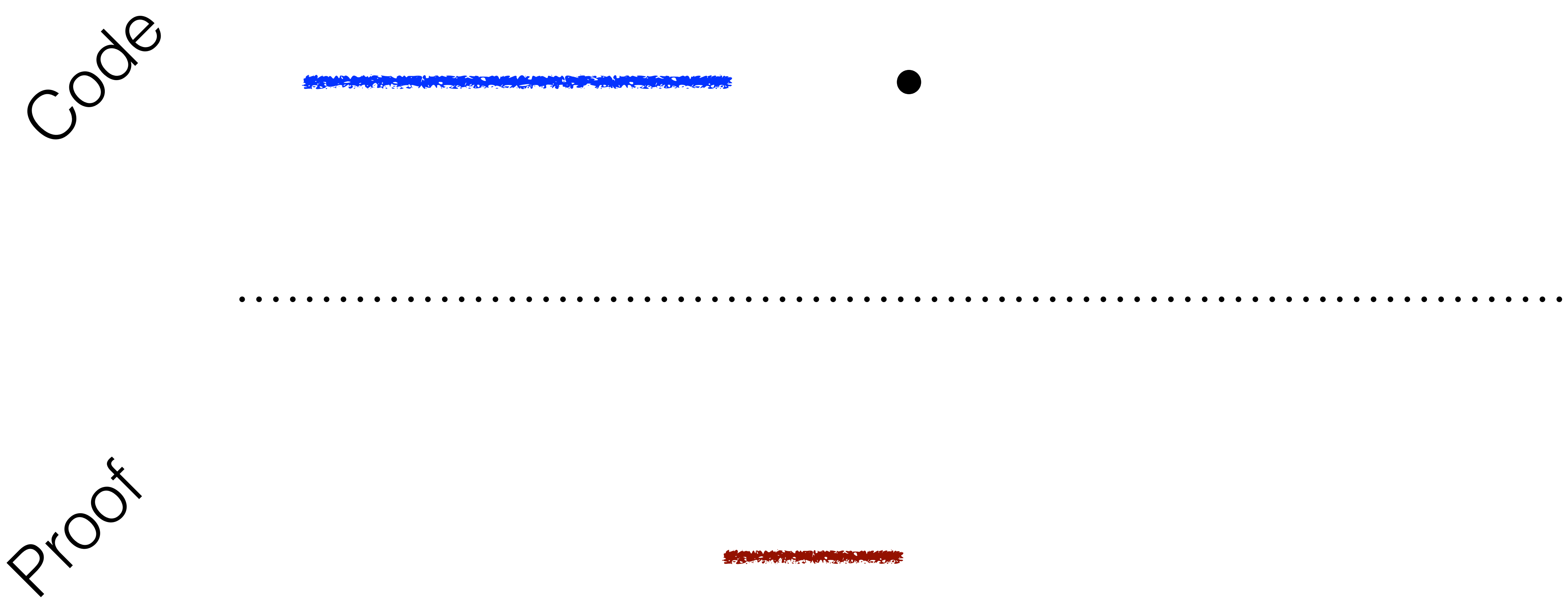
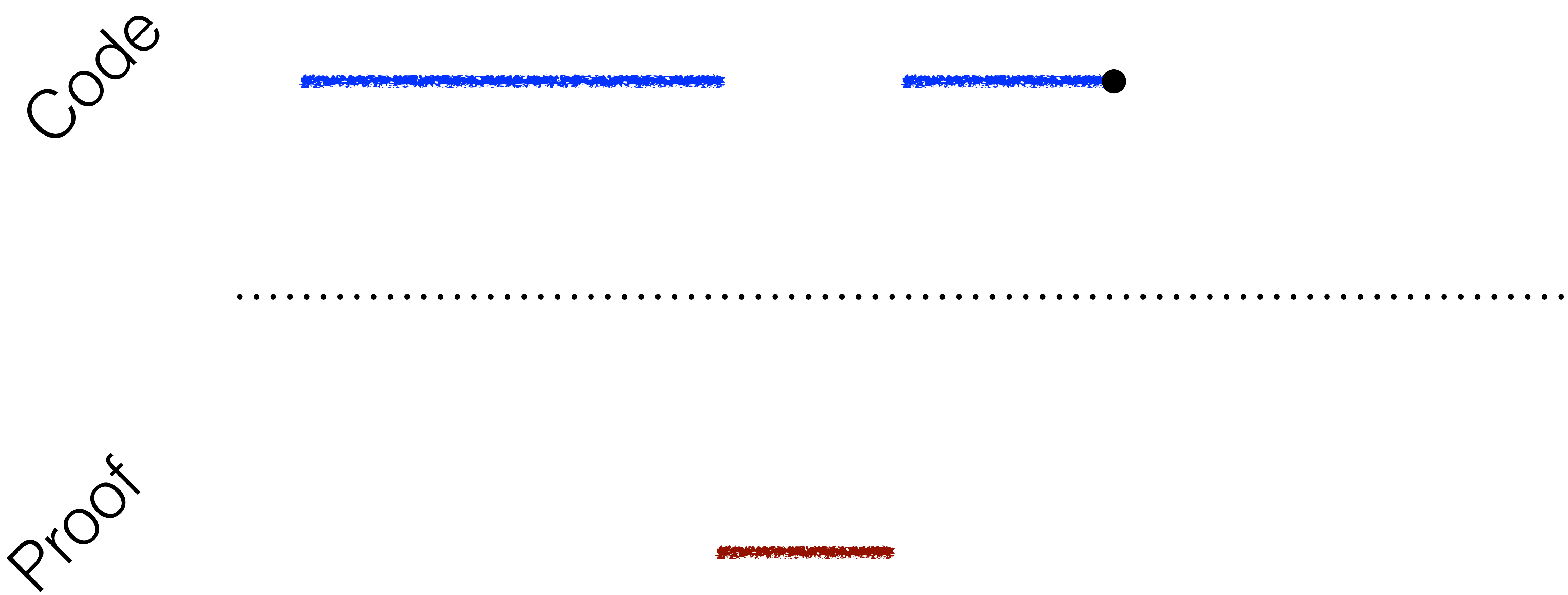
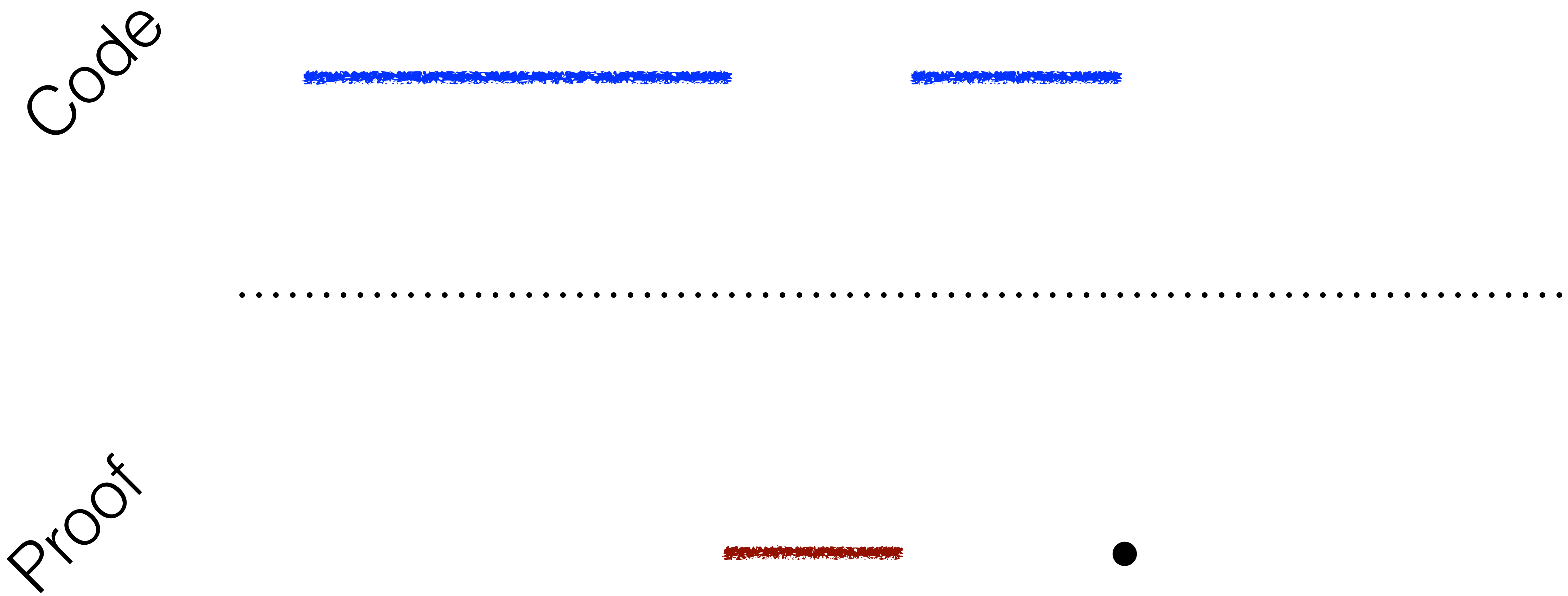

Code
Proof


Code
Proof


Code
Proof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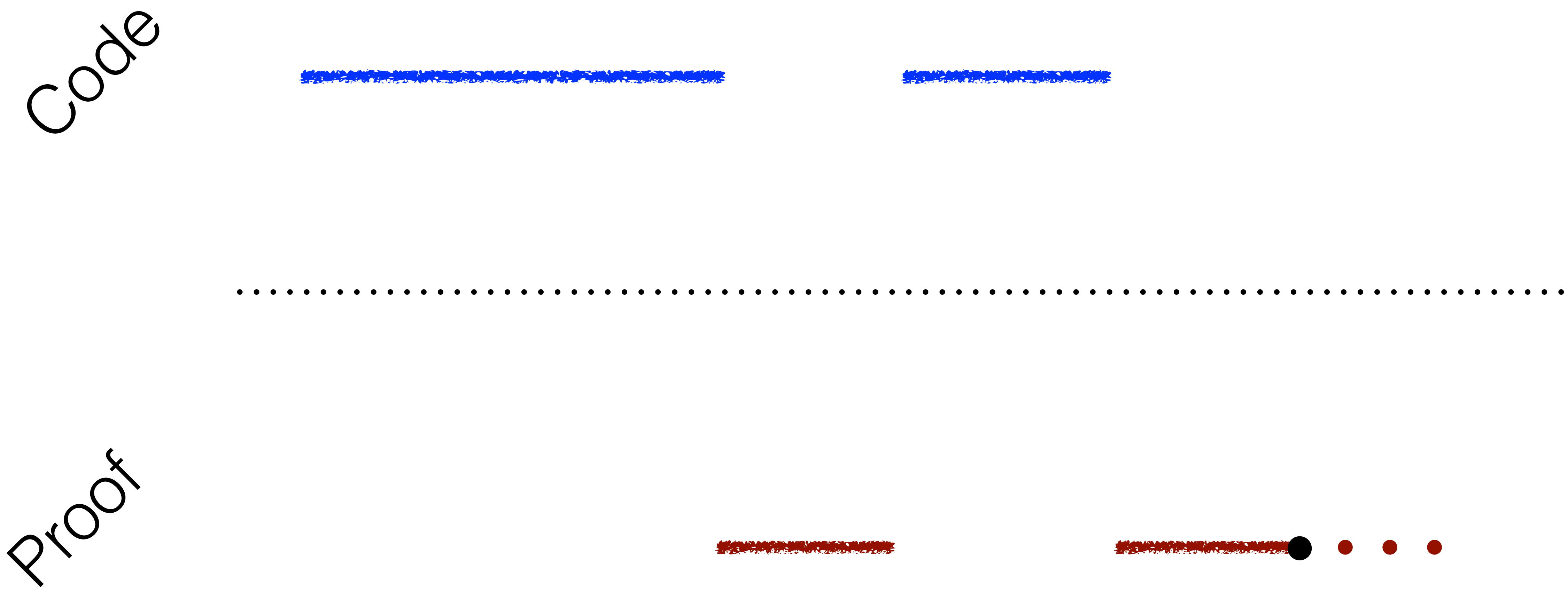
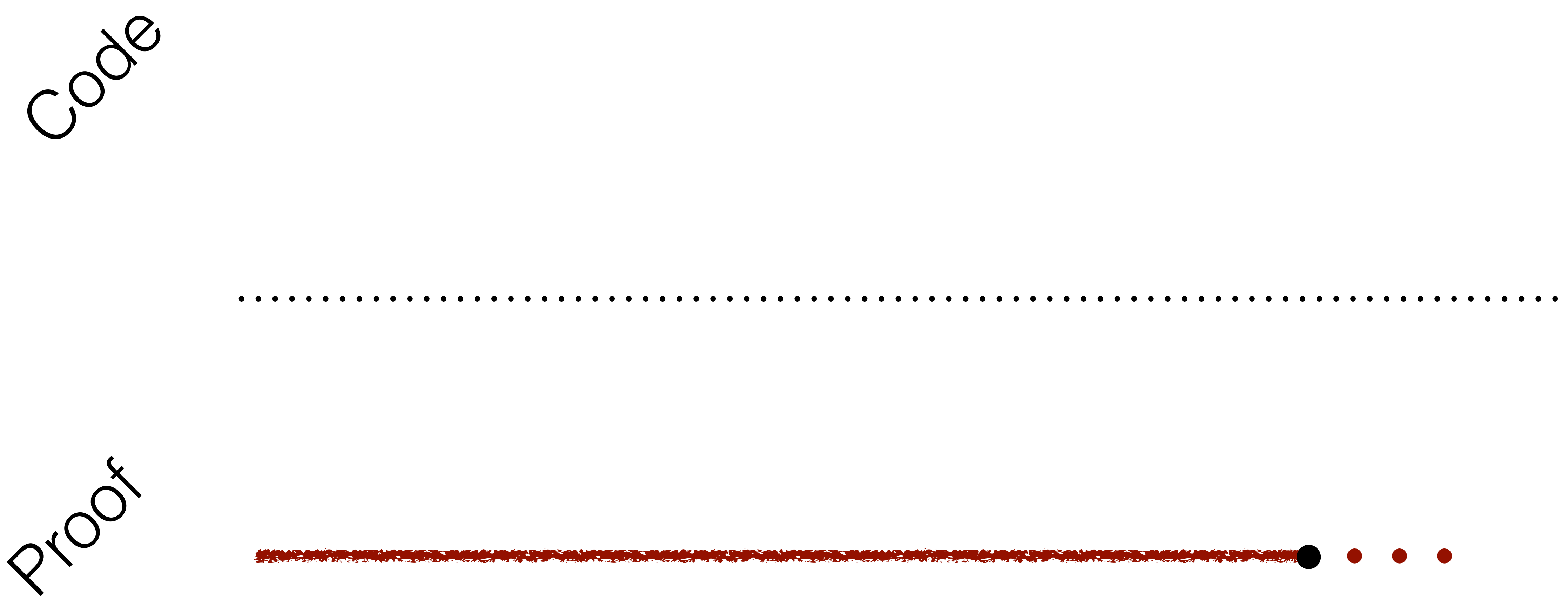

Code
Proof

Just write the proof

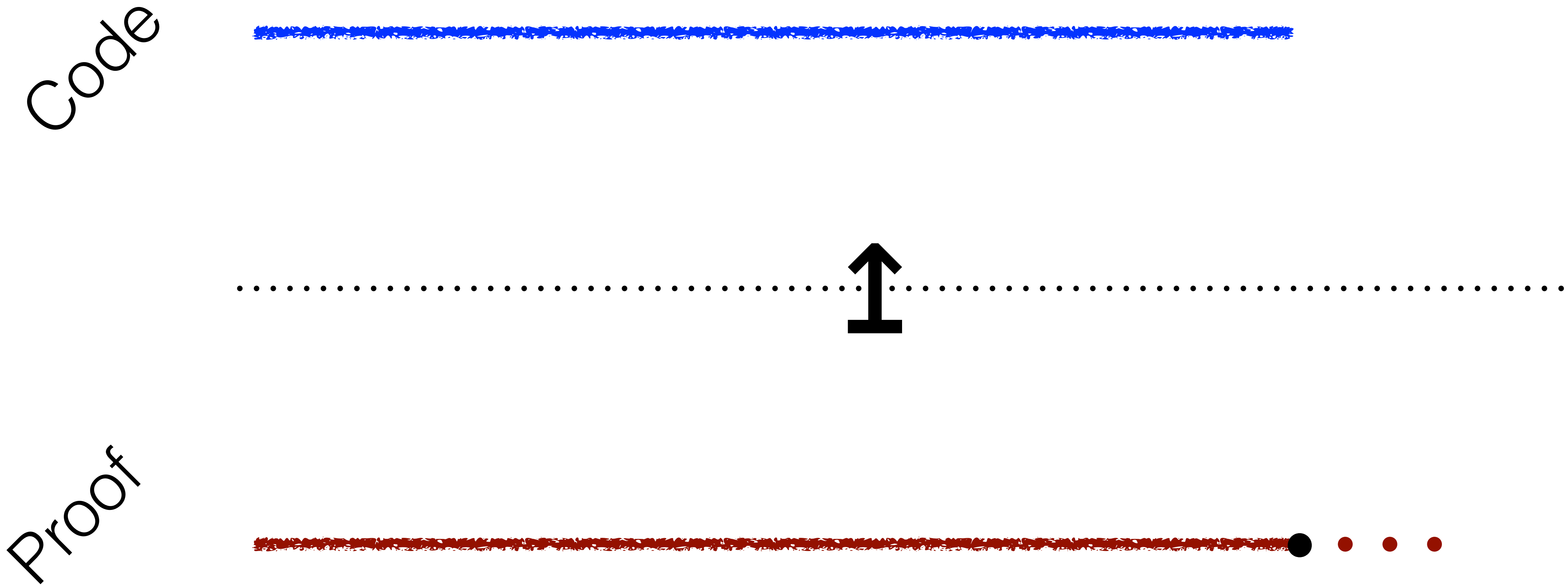
Code

Proof

-



Code
Proof


Code
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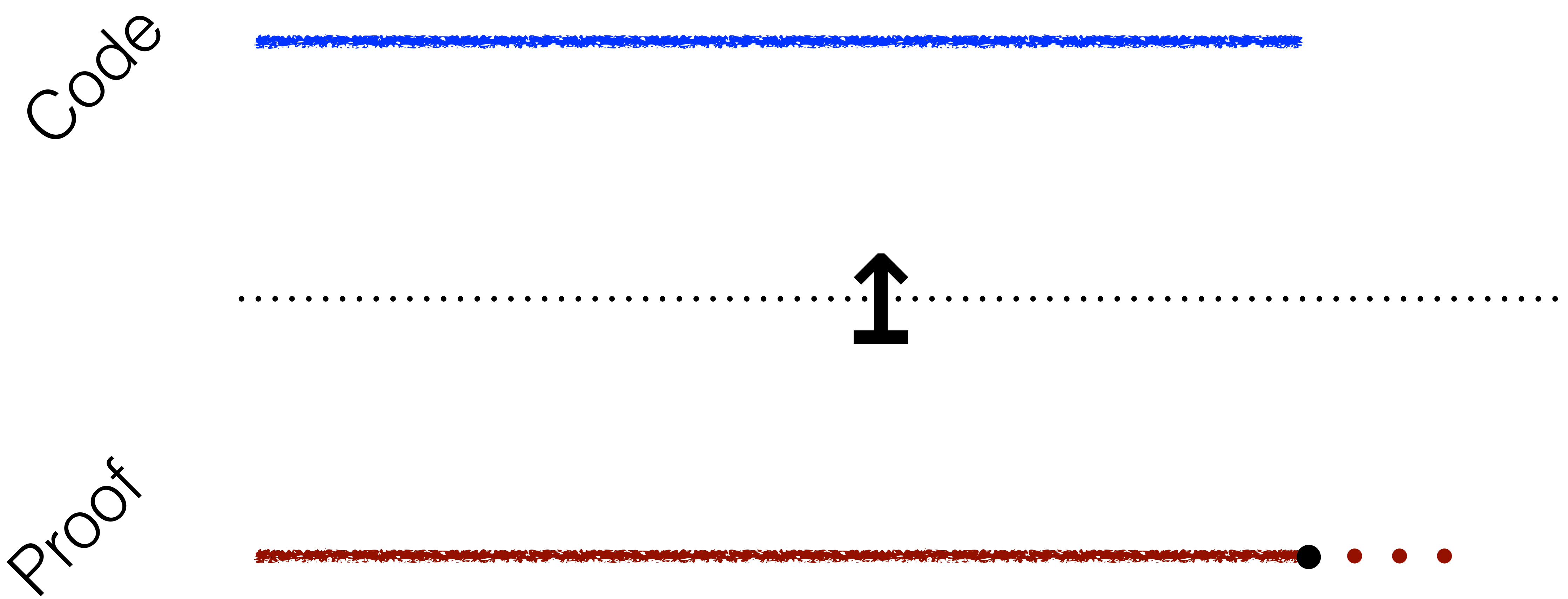
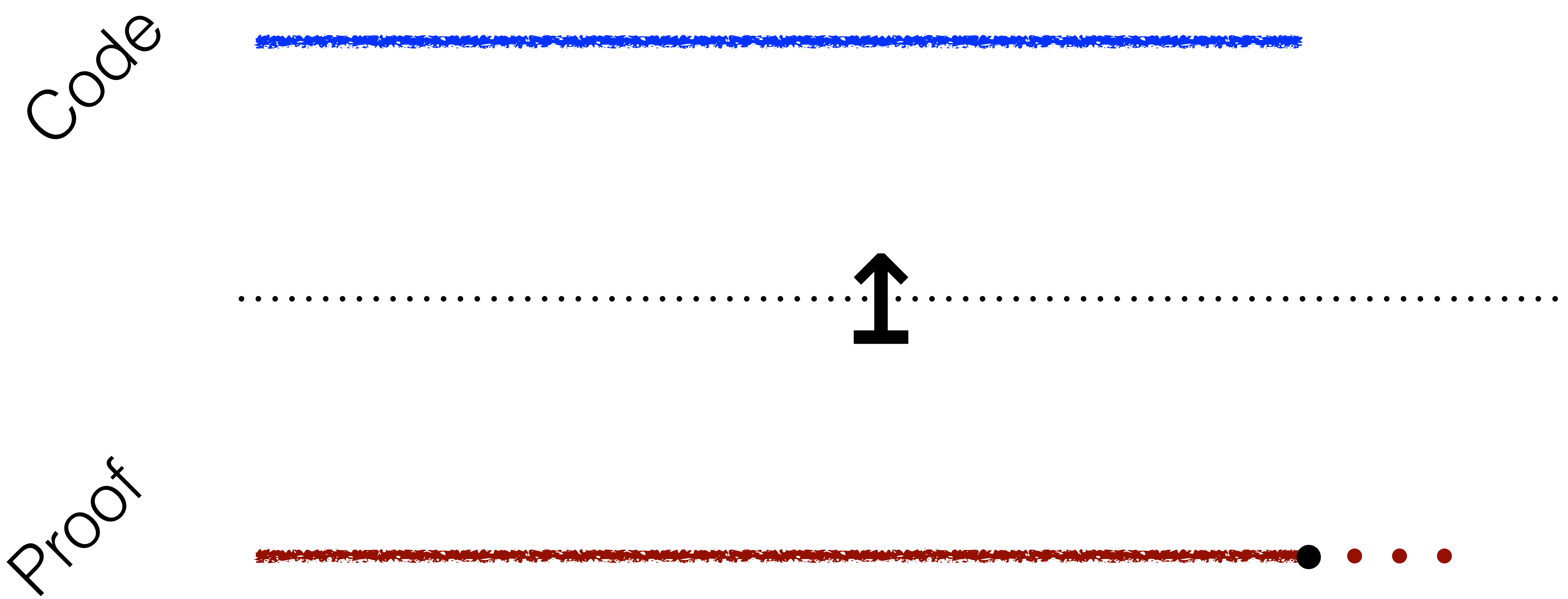


- Calculational Abstract Interpretation (spec $\mapsto$ alg)



- Calculational Abstract Interpretation (spec ↦ alg)
- Constructive Logic (proof ↦ code)

Abstract Interpretation and Constructive Logic don’t mix

(until now)

Three Stories

Three Stories

Direct Verification

✗ `framework`
✓ `mechanize`

Three Stories

Direct Verification	✗ `framework` ✓ `mechanize`
Abstract Interpretation	✓ `framework` ✗ `mechanize`

Three Stories

Direct Verification	✗ `framework` ✓ `mechanize`
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Constructive GCs	✓ `framework` ✓ `mechanize`

Direct Verification

Direct Verification

```
succ  : ℕ → ℕ
```

Direct Verification

$\texttt{succ} : \mathbb{N} \to \mathbb{N}$

$\mathbb{P} \coloneqq \{\mathtt{E}, \mathtt{O}\}$

Direct Verification

$\mathtt{succ} : \mathbb{N} \to \mathbb{N}$

$\mathbb{P} \coloneqq \{\mathtt{E}, \mathtt{O}\}$

$\mathtt{succ}^\sharp : \mathbb{P} \to \mathbb{P}$
$\mathtt{succ}^\sharp(\mathtt{E}) \coloneqq \mathtt{O}$
$\mathtt{succ}^\sharp(\mathtt{O}) \coloneqq \mathtt{E}$

Direct Verification

$$\mathtt{succ} : \mathbb{N} \to \mathbb{N}$$

$$\mathbb{P} \coloneqq \{E, O\}$$

$$\begin{aligned} \mathtt{succ}^\sharp &: \mathbb{P} \to \mathbb{P} \\ \mathtt{succ}^\sharp(E) &\coloneqq O \\ \mathtt{succ}^\sharp(O) &\coloneqq E \end{aligned}$$

$$\begin{aligned} [\![\_]\!] &: \mathbb{P} \to \wp(\mathbb{N}) \\ [\![E]\!] &\coloneqq \{\, n \mid \mathrm{even}(n) \,\} \\ [\![O]\!] &\coloneqq \{\, n \mid \mathrm{odd}(n) \,\} \end{aligned}$$

Direct Verification

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&[\![_]\!] : \mathbb{P} \to \wp(\mathbb{N}) \\
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&[\![\mathtt{O}]\!] \coloneqq \{\, n \mid \mathtt{odd}(n) \,\}
\end{aligned}$$

$$\boxed{\mathtt{sound} : n \in [\![p]\!] \implies \mathtt{succ}(n) \in [\![\mathtt{succ}\sharp(p)]\!]}$$

Direct Verification

```
succ♯ : ℙ → ℙ
succ♯(E) ≔ O
succ♯(O) ≔ E
```

```
⟦_⟧ : ℙ → ℘(ℕ)
⟦E⟧ ≔ { n | even(n) }
⟦O⟧ ≔ { n |  odd(n) }
```

Direct Verification

$$\wp(\mathbb{N}) \coloneqq \mathbb{N} \to \texttt{prop}$$

$$\begin{aligned}
&\texttt{succ}\sharp : \mathbb{P} \to \mathbb{P} \\
&\texttt{succ}\sharp(\texttt{E}) \coloneqq \texttt{O} \\
&\texttt{succ}\sharp(\texttt{O}) \coloneqq \texttt{E}
\end{aligned}$$

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&[\![\texttt{E}]\!] \coloneqq \{\ n \mid \texttt{even}(n)\ \} \\
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Direct Verification

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$[\![\_]\!] : \mathbb{P} \rightarrow (\mathbb{N} \rightarrow \texttt{prop})$
$[\![\texttt{E}]\!] \coloneqq \{\ n \mid \texttt{even}(n)\ \}$
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succ$\sharp$ can be extracted and executed

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$[\![\_]\!]$ can be defined constructively without axioms

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$\textbf{succ}\sharp$ can be extracted and executed

$[\![\_]\!]$ can be defined constructively without axioms

Is $\textbf{succ}\sharp$ optimal? (why not $\texttt{succ}\sharp(\texttt{E}) \coloneqq \{\texttt{E},\texttt{O}\}$)

Direct Verification

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$[\![\_]\!] : \mathbb{P} \rightarrow (\mathbb{N} \rightarrow \texttt{prop})$
$[\![\texttt{E}]\!] \coloneqq \texttt{even}$
$[\![\texttt{O}]\!] \coloneqq \texttt{odd}$

succ♯ can be extracted and executed

$[\![\_]\!]$ can be defined constructively without axioms

Is **succ**♯ optimal? (why not $\texttt{succ}\sharp(\texttt{E}) \coloneqq \{\texttt{E},\texttt{O}\}$)

Can one derive **succ**♯ from **succ** directly?

Direct Verification

$\wp(\mathbb{N}) \coloneqq \mathbb{N} \rightarrow \texttt{prop}$

$\texttt{succ}^\sharp : \mathbb{P} \rightarrow \mathbb{P}$
$\texttt{succ}^\sharp(\texttt{E}) \coloneqq \texttt{O}$
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$\mathbf{succ}^\sharp$ can be extracted and executed

$[\![\_]\!]$ can be defined constructively without axioms

✓ mechanize

Is $\mathbf{succ}^\sharp$ optimal? (why not $\texttt{succ}^\sharp(\texttt{E}) \coloneqq \{\texttt{E},\texttt{O}\}$)

Can one derive $\mathbf{succ}^\sharp$ from **succ** directly?

✗ framework

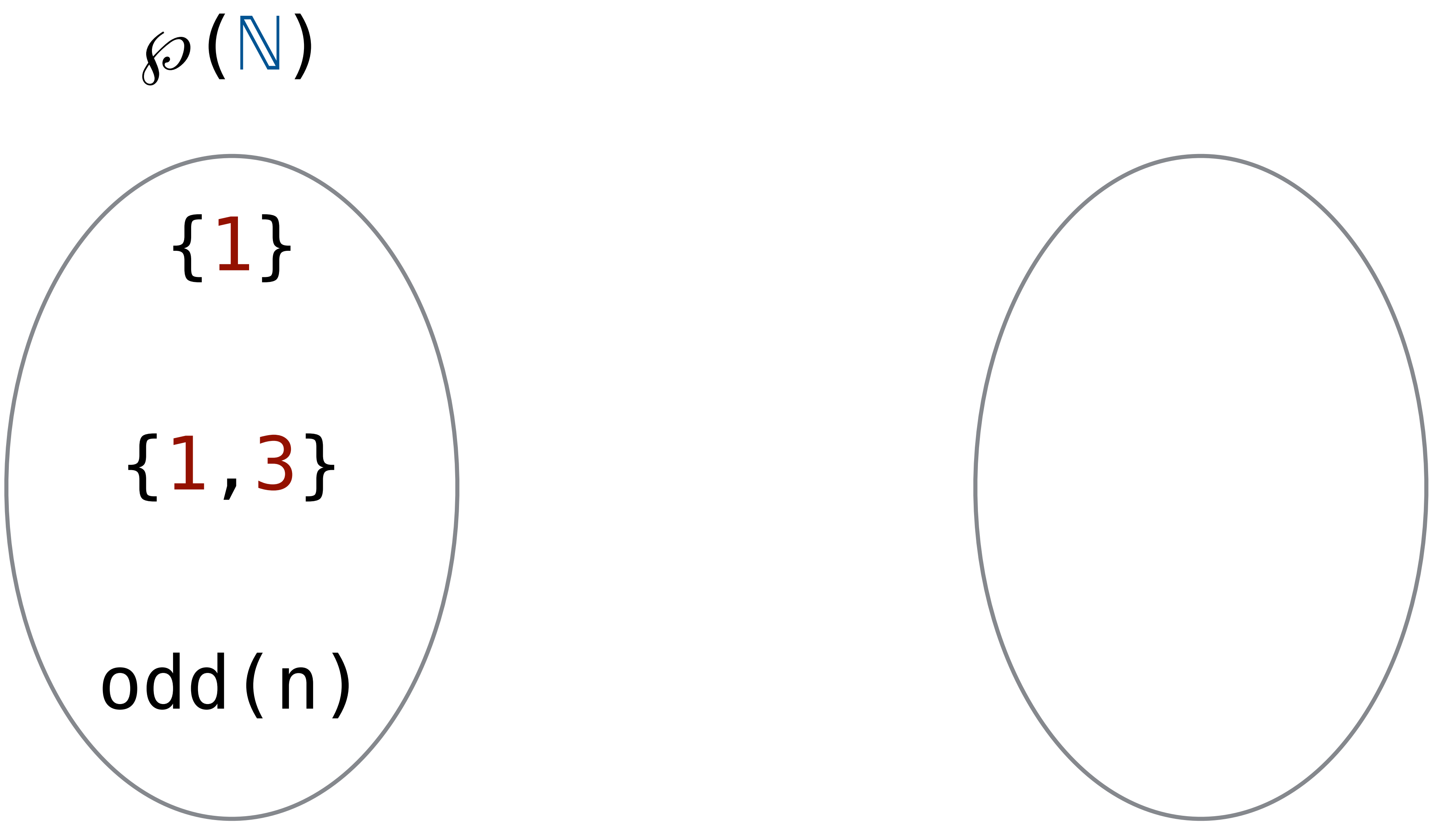
Three Stories

Direct Verification	✗ `framework` ✓ `mechanize`
Abstract Interpretation	✓ `framework` ✗ `mechanize`
Constructive GCs	✓ `framework` ✓ `mechanize`

Abstract Interpretation

Abstract Interpretation

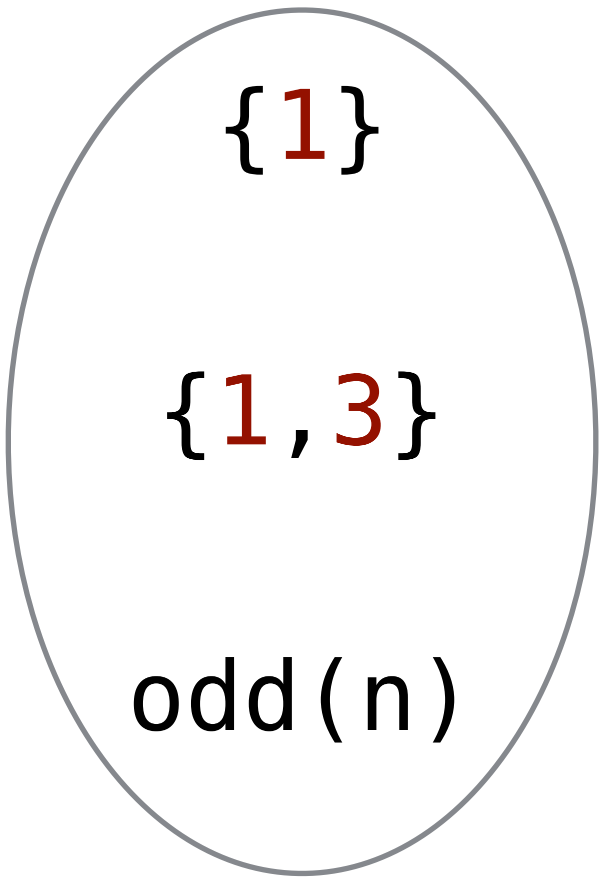
Abstract Interpretation

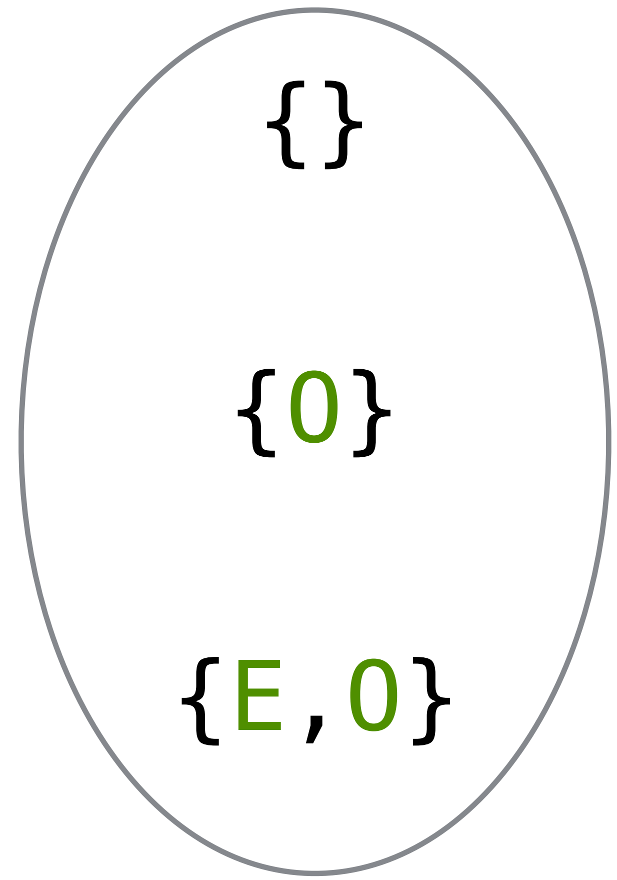


Abstract Interpretation

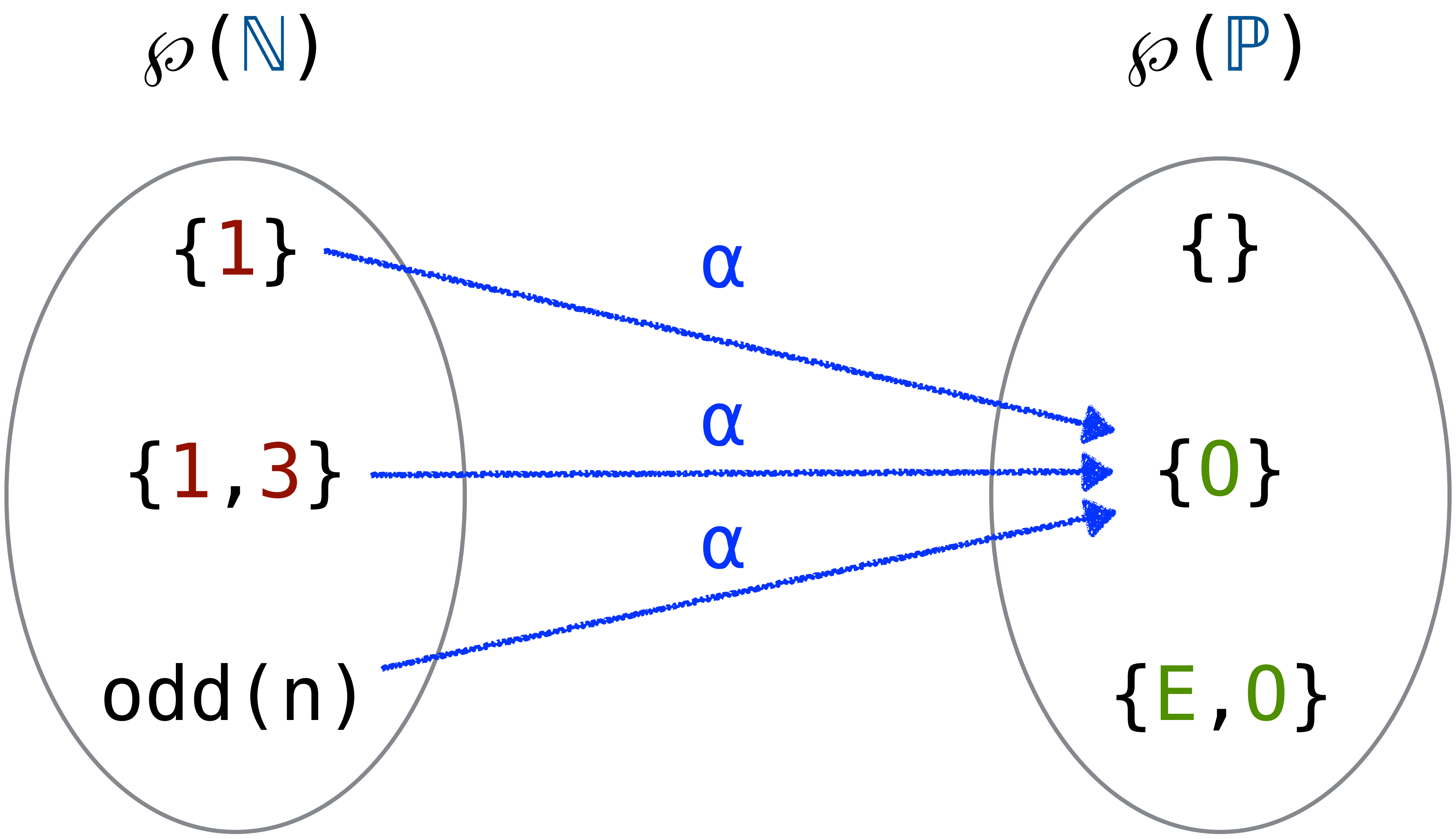
$\wp(\mathbb{N})$

$\wp(\mathbb{P})$

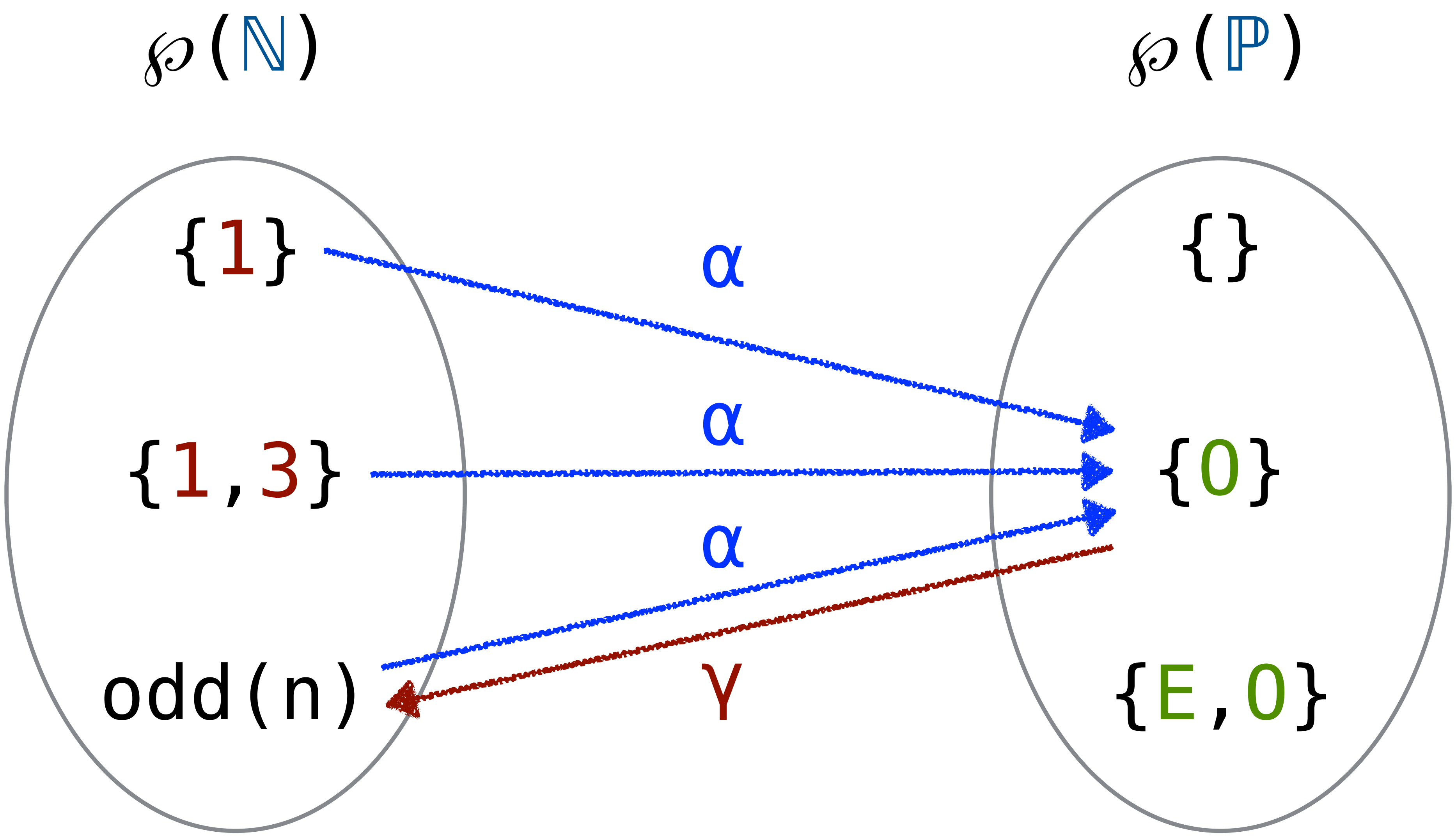




Abstract Interpretation



Abstract Interpretation



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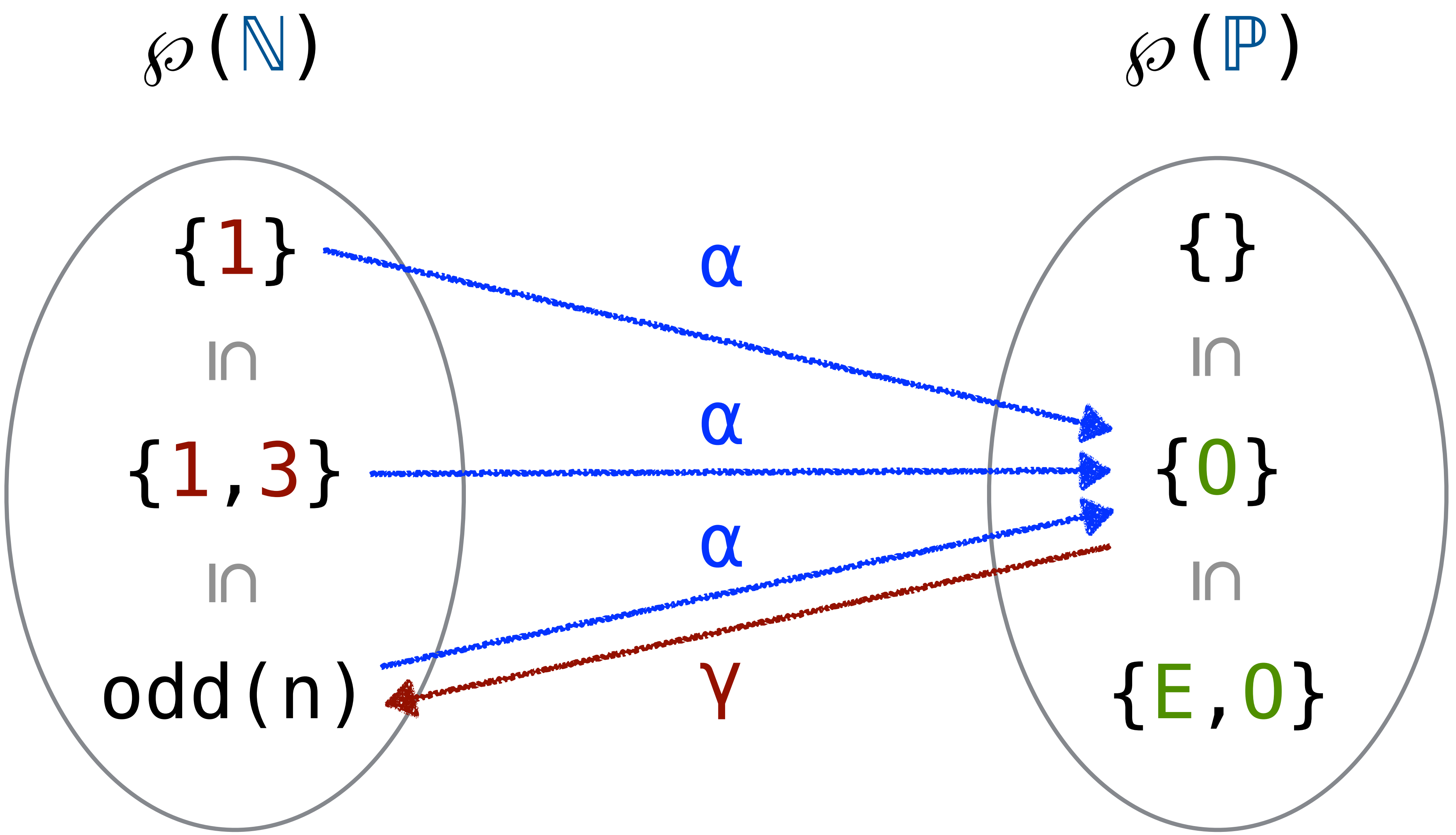
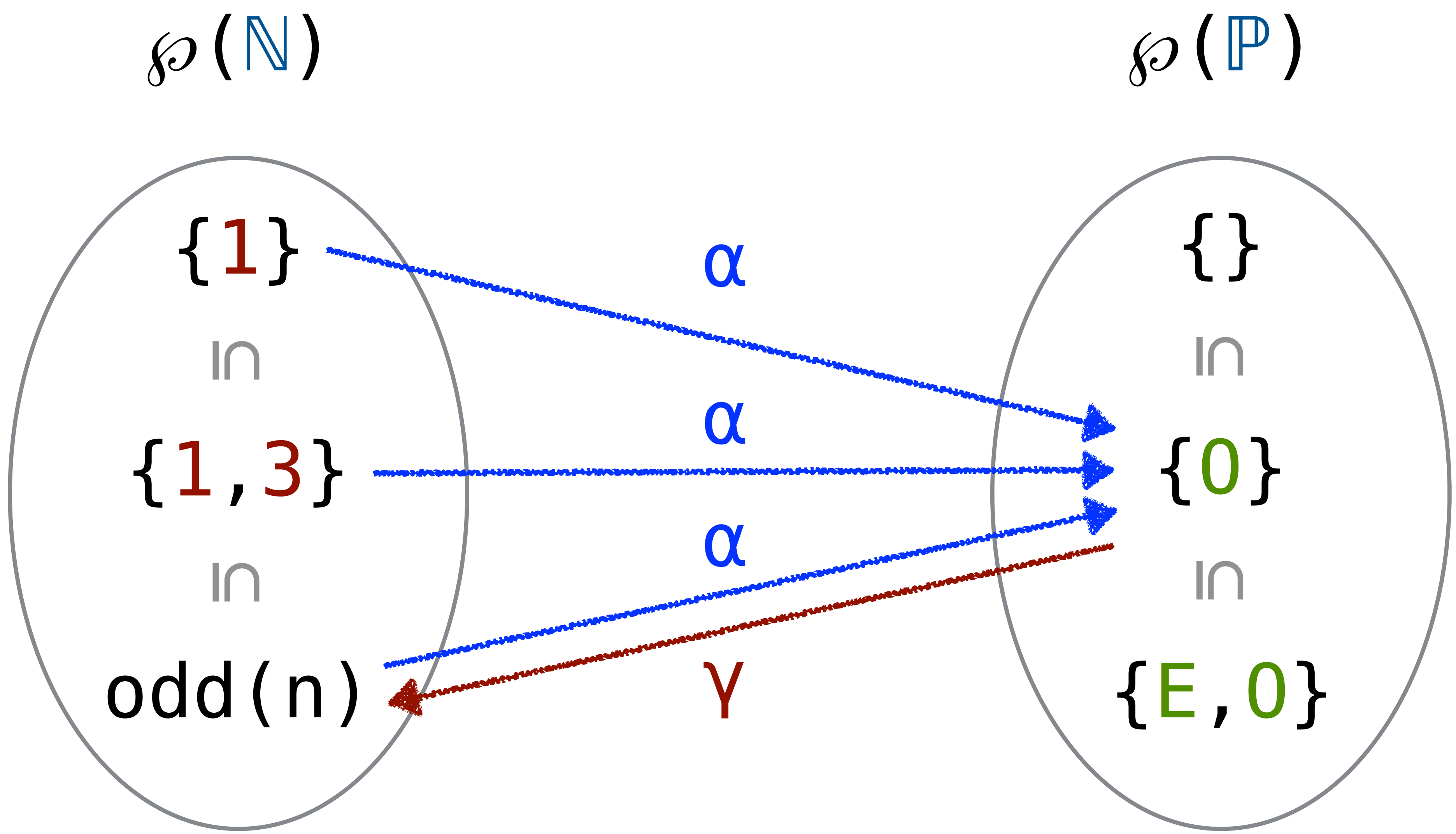
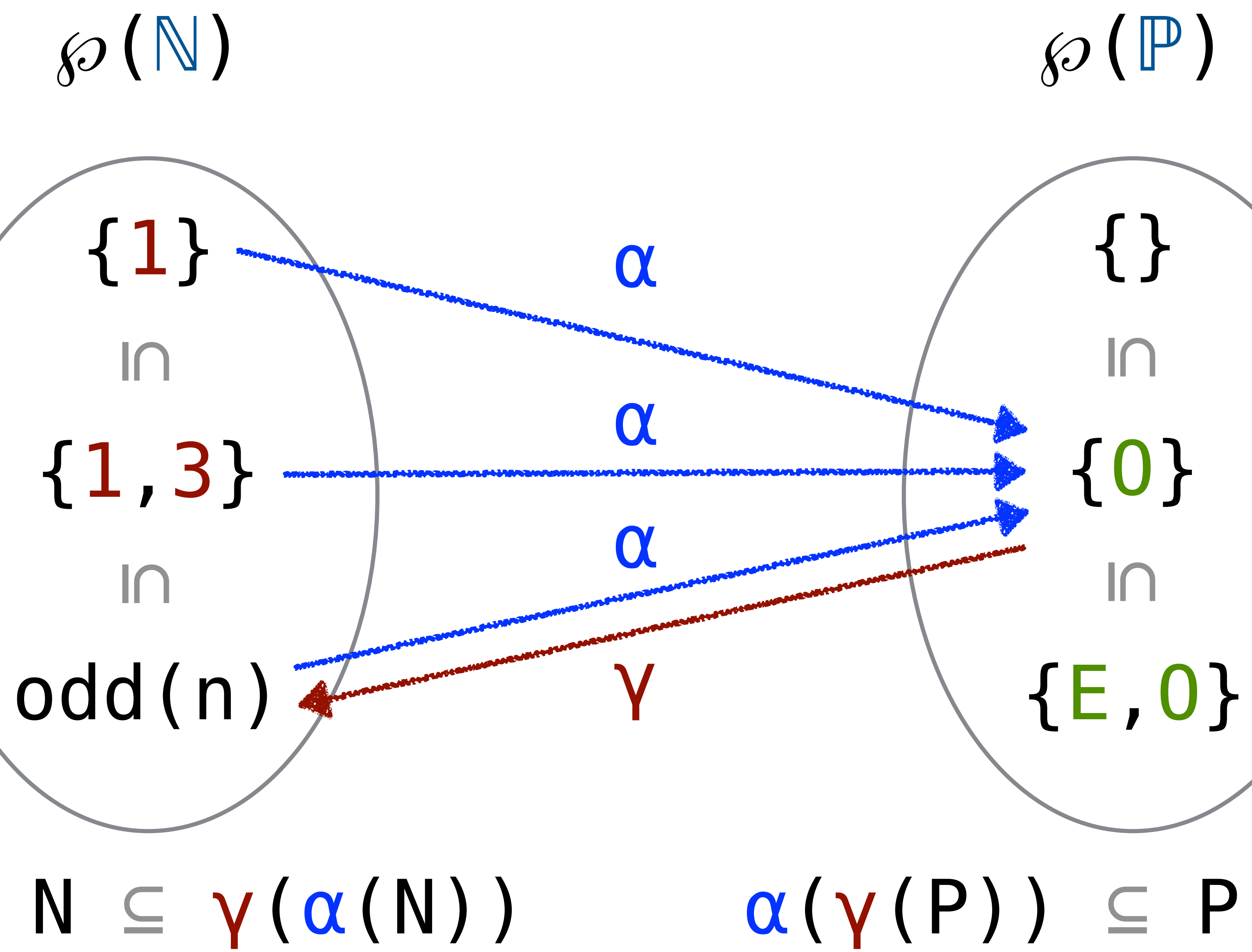


Abstract Interpretation

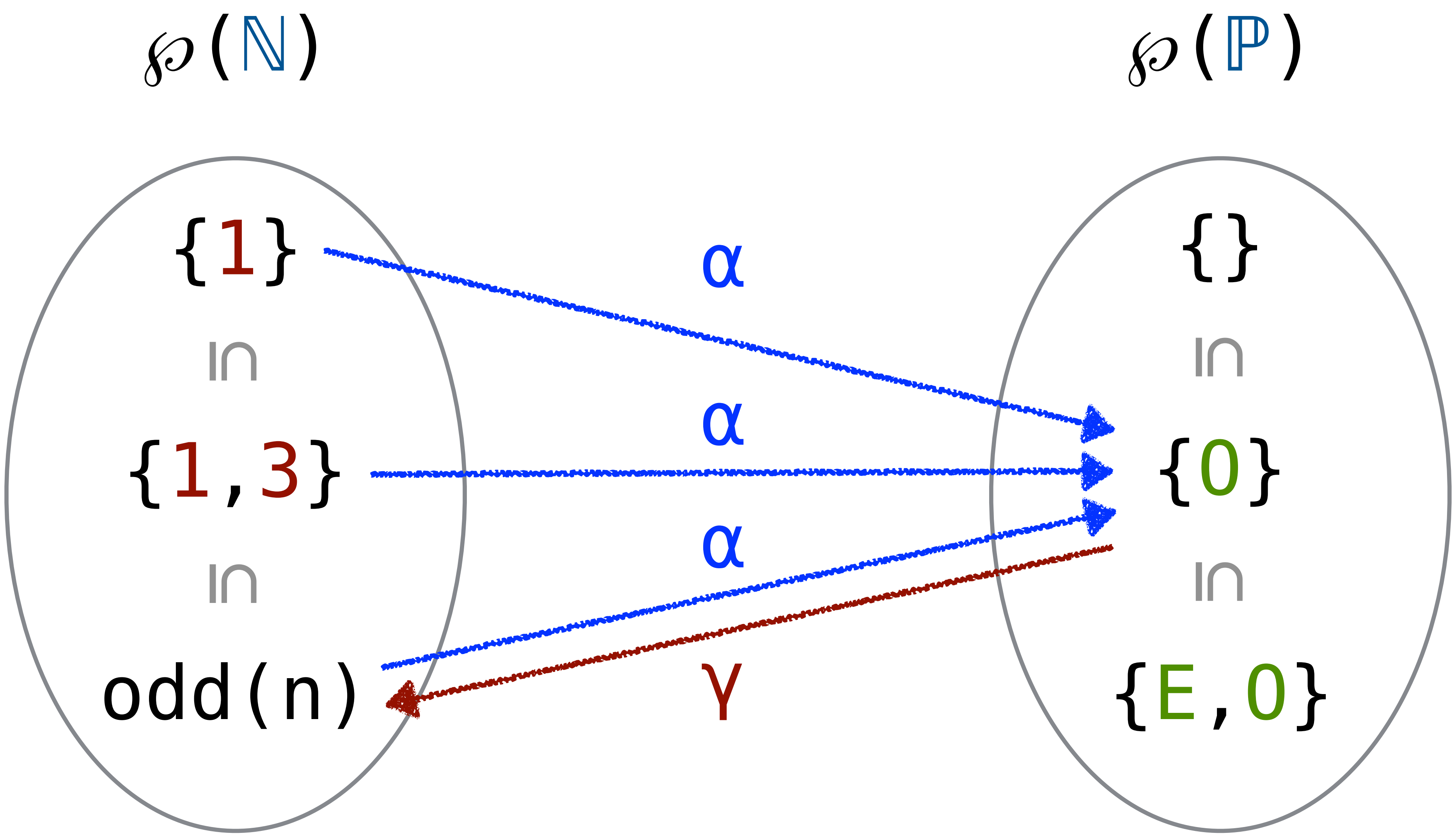


$$N \subseteq \gamma(\alpha(N))$$

Abstract Interpretation



Abstract Interpretation



$$\frac{N \subseteq \gamma(\alpha(N)) \quad \wedge \quad \alpha(\gamma(P)) \subseteq P}{N \subseteq \gamma(P) \;\Leftrightarrow\; \alpha(N) \subseteq P}$$

Abstract Interpretation

$$\texttt{N} \in \wp(\mathbb{N})$$
$$\texttt{P} \in \wp(\mathbb{N})$$

"$\texttt{P}$ is sound for $\texttt{N}$"

$$\alpha(\texttt{N}) \subseteq \texttt{P}$$

Abstract Interpretation

$\mathtt{N} \in \wp(\mathbb{N})$

$\mathtt{P} \in \wp(\mathbb{N})$

"$\mathtt{P}$ is sound for $\mathtt{N}$"

$\alpha(\mathtt{N}) \subseteq \mathtt{P}$

$\mathtt{f}^{\mathtt{N}} \in \wp(\mathbb{N}) \to \wp(\mathbb{N})$

$\mathtt{f}^{\mathtt{P}} \in \wp(\mathbb{P}) \to \wp(\mathbb{P})$

"$\mathtt{f}^{\mathtt{p}}$ is sound for $\mathtt{f}^{\mathtt{N}}$"

Abstract Interpretation

$\mathtt{N} \in \wp(\mathbb{N})$

$\mathtt{P} \in \wp(\mathbb{N})$

"$\mathtt{P}$ is sound for $\mathtt{N}$"

$\alpha(\mathtt{N}) \subseteq \mathtt{P}$

$\mathtt{f}^{\mathtt{N}} \in \wp(\mathbb{N}) \to \wp(\mathbb{N})$

$\mathtt{f}^{\mathtt{P}} \in \wp(\mathbb{P}) \to \wp(\mathbb{P})$

"$\mathtt{f}^{\mathtt{p}}$ is sound for $\mathtt{f}^{\mathtt{N}}$"

$\vec{\alpha}(\mathtt{f}^{\mathtt{N}}) \subseteq \mathtt{f}^{\mathtt{P}}$

Abstract Interpretation

$\mathtt{N} \in \wp(\mathbb{N})$

$\mathtt{P} \in \wp(\mathbb{N})$

"$\mathtt{P}$ is sound for $\mathtt{N}$"

$\alpha(\mathtt{N}) \subseteq \mathtt{P}$

$\mathtt{f}^{\mathtt{N}} \in \wp(\mathbb{N}) \to \wp(\mathbb{N})$

$\mathtt{f}^{\mathtt{P}} \in \wp(\mathbb{P}) \to \wp(\mathbb{P})$

"$\mathtt{f}^{\mathtt{p}}$ is sound for $\mathtt{f}^{\mathtt{N}}$"

$\alpha \circ \mathtt{f}^{\mathtt{N}} \circ \gamma \subseteq \mathtt{f}^{\mathtt{P}}$

Abstract Interpretation

$\alpha \ : \ \wp(\mathbb{N}) \rightarrow \wp(\mathbb{P})$

$\gamma \ : \ \wp(\mathbb{P}) \rightarrow \wp(\mathbb{N})$

Abstract Interpretation

$$\alpha : \wp(\mathbb{N}) \to \wp(\mathbb{P})$$
$$\alpha(N) \coloneqq \{\text{parity}(n) \mid n \in N\}$$

$$\gamma : \wp(\mathbb{P}) \to \wp(\mathbb{N})$$

Abstract Interpretation

$\alpha : \wp(\mathbb{N}) \to \wp(\mathbb{P})$

$\alpha(N) \coloneqq \{parity(n) \mid n \in N\}$

e.g. $\alpha(\{1,3\}) = \{0\}$

$\gamma : \wp(\mathbb{P}) \to \wp(\mathbb{N})$

Abstract Interpretation

$$\alpha : \wp(\mathbb{N}) \to \wp(\mathbb{P})$$
$$\alpha(N) \coloneqq \{\mathrm{parity}(n) \mid n \in N\}$$

e.g. $\alpha(\{1,3\}) = \{0\}$

$$\gamma : \wp(\mathbb{P}) \to \wp(\mathbb{N})$$
$$\gamma(P) \coloneqq \{n \mid p \in P \land n \in [\![p]\!]\}$$

Abstract Interpretation

$$\alpha : \wp(\mathbb{N}) \to \wp(\mathbb{P})$$
$$\alpha(N) \coloneqq \{\mathrm{parity}(n) \mid n \in N\}$$

e.g. $\alpha(\{1,3\}) = \{0\}$

$$\gamma : \wp(\mathbb{P}) \to \wp(\mathbb{N})$$
$$\gamma(P) \coloneqq \{n \mid p \in P \land n \in [\![p]\!]\}$$

e.g. $\gamma(\{0\}) = \{n \mid \mathrm{odd}(n)\}$

Abstract Interpretation

$$\alpha \ : \ \wp(\mathbb{N}) \rightarrow \wp(\mathbb{P})$$
$$\gamma \ : \ \wp(\mathbb{P}) \rightarrow \wp(\mathbb{N})$$

Abstract Interpretation

$$\alpha : \wp(\mathbb{N}) \to \wp(\mathbb{P})$$
$$\gamma : \wp(\mathbb{P}) \to \wp(\mathbb{N})$$

$$\uparrow\mathtt{succ} : \wp(\mathbb{N}) \to \wp(\mathbb{N})$$
$$\uparrow\mathtt{succ}(N) \coloneqq \{\mathtt{succ}(n) \mid n \in N\}$$

$$\uparrow\mathtt{succ}^\sharp : \wp(\mathbb{P}) \to \wp(\mathbb{P})$$
$$\uparrow\mathtt{succ}^\sharp(P) = \{\mathtt{succ}^\sharp(p) \mid p \in P\}$$

Abstract Interpretation

$$\alpha : \wp(\mathbb{N}) \to \wp(\mathbb{P})$$
$$\gamma : \wp(\mathbb{P}) \to \wp(\mathbb{N})$$

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$$\uparrow\text{succ}(N) \coloneqq \{\text{succ}(n) \mid n \in N\}$$

$$\uparrow\text{succ}\sharp : \wp(\mathbb{P}) \to \wp(\mathbb{P})$$
$$\uparrow\text{succ}\sharp(P) = \{\text{succ}\sharp(p) \mid p \in P\}$$

$$\boxed{\text{sound} : \alpha(\uparrow\text{succ}(\gamma(P))) \subseteq \uparrow\text{succ}\sharp(P)}$$

Abstract Interpretation

$$\alpha : \wp(\mathbb{N}) \to \wp(\mathbb{P})$$
$$\gamma : \wp(\mathbb{P}) \to \wp(\mathbb{N})$$

$$\uparrow\texttt{succ} : \wp(\mathbb{N}) \to \wp(\mathbb{N})$$
$$\uparrow\texttt{succ}(N) \coloneqq \{\texttt{succ}(n) \mid n \in N\}$$

$$\uparrow\texttt{succ}^\sharp : \wp(\mathbb{P}) \to \wp(\mathbb{P})$$
$$\uparrow\texttt{succ}^\sharp(P) = \{\texttt{succ}^\sharp(p) \mid p \in P\}$$

sound	$\alpha(\uparrow\texttt{succ}(\gamma(P))) \subseteq \uparrow\texttt{succ}^\sharp(P)$
optimal	$\alpha(\uparrow\texttt{succ}(\gamma(P))) = \uparrow\texttt{succ}^\sharp(P)$

Abstract Interpretation

$$\boxed{\texttt{optimal : } \alpha(\uparrow\texttt{succ}(\gamma(\texttt{P}))) = \uparrow\texttt{succ}^\sharp(\texttt{P})}$$

Abstract Interpretation

```
calc : α(↑succ(γ(P))) = … ≜ ↑succ♯(P)
```

Abstract Interpretation

$$\boxed{\texttt{calc : } \alpha(\uparrow\texttt{succ}(\gamma(\texttt{P}))) = \ldots \triangleq \uparrow\texttt{succ}^\sharp(\texttt{P})}$$

$$\alpha(\uparrow\texttt{succ}(\gamma(\{\texttt{E}\})))$$

Abstract Interpretation

$\boxed{\texttt{calc : } \alpha(\uparrow\texttt{succ}(\gamma(\texttt{P}))) = \ldots \triangleq \uparrow\texttt{succ}^\sharp(\texttt{P})}$

$$\alpha(\uparrow\texttt{succ}(\gamma(\{\texttt{E}\})))$$
$$= \alpha(\uparrow\texttt{succ}(\{n \mid \texttt{even}(n)\}))$$

Abstract Interpretation

$\boxed{\texttt{calc : } \alpha(\uparrow\texttt{succ}(\gamma(\texttt{P}))) = \ldots \triangleq \uparrow\texttt{succ}^{\sharp}(\texttt{P})}$

$$
\begin{aligned}
&\alpha(\uparrow\mathrm{succ}(\gamma(\{E\}))) \\
&= \alpha(\uparrow\mathrm{succ}(\{n \mid \mathrm{even}(n)\})) \\
&= \alpha(\{\mathrm{succ}(n) \mid \mathrm{even}(n)\})
\end{aligned}
$$

Abstract Interpretation

$\boxed{\texttt{calc : } \alpha(\uparrow\mathrm{succ}(\gamma(P))) = \ldots \triangleq \uparrow\mathrm{succ}^\sharp(P)}$

$$
\begin{aligned}
&\alpha(\uparrow\mathrm{succ}(\gamma(\{E\}))) \\
&= \alpha(\uparrow\mathrm{succ}(\{n \mid \mathrm{even}(n)\})) \\
&= \alpha(\{\mathrm{succ}(n) \mid \mathrm{even}(n)\}) \\
&= \alpha(\{n \mid \mathrm{odd}(n)\})
\end{aligned}
$$

Abstract Interpretation

$\boxed{\texttt{calc} : \alpha(\uparrow\mathrm{succ}(\gamma(P))) = \ldots \triangleq \uparrow\mathrm{succ}^\sharp(P)}$

$$
\begin{aligned}
&\alpha(\uparrow\mathrm{succ}(\gamma(\{E\}))) \\
&= \alpha(\uparrow\mathrm{succ}(\{n \mid \mathrm{even}(n)\})) \\
&= \alpha(\{\mathrm{succ}(n) \mid \mathrm{even}(n)\}) \\
&= \alpha(\{n \mid \mathrm{odd}(n)\}) \\
&= \{0\}
\end{aligned}
$$

Abstract Interpretation

```
calc : α(↑succ(γ(P))) = … ≜ ↑succ♯(P)
```

```
α(↑succ(γ({E})))
= α(↑succ({n | even(n)}))
= α({succ(n) | even(n)})
= α({n | odd(n)})
= {O}
≜ ↑succ♯({E})
```

Abstract Interpretation

$$\boxed{\texttt{calc : } \alpha(\uparrow\texttt{succ}(\gamma(\texttt{P}))) = \ldots \triangleq \uparrow\texttt{succ}^\sharp(\texttt{P})}$$

$$
\begin{aligned}
& \alpha(\uparrow\texttt{succ}(\gamma(\{\texttt{E}\}))) \\
& = \alpha(\uparrow\texttt{succ}(\{\texttt{n} \mid \texttt{even}(\texttt{n})\})) \\
& = \alpha(\{\texttt{succ}(\texttt{n}) \mid \texttt{even}(\texttt{n})\}) \\
& = \alpha(\{\texttt{n} \mid \texttt{odd}(\texttt{n})\}) \\
& = \{\texttt{O}\} \\
& \triangleq \uparrow\texttt{succ}^\sharp(\{\texttt{E}\})
\end{aligned}
$$

[CDGAI: Cousot 1999]

Abstract Interpretation

$$\boxed{\texttt{calc : } \alpha(\uparrow\texttt{succ}(\gamma(\texttt{P}))) = \ldots \triangleq \uparrow\texttt{succ}^\sharp(\texttt{P})}$$

$$\uparrow\texttt{succ}^\sharp(\{\texttt{E}\}) \triangleq \{\texttt{O}\}$$

Abstract Interpretation

$$\wp(\mathbb{P}) \quad \mapsto \quad \ldots \quad \mapsto \quad \wp(\mathbb{P})$$

$$\boxed{\texttt{calc : } \alpha(\uparrow\texttt{succ}(\gamma(\texttt{P}))) = \ldots \triangleq \uparrow\texttt{succ}^\sharp(\texttt{P})}$$

$$\uparrow\texttt{succ}^\sharp(\{\texttt{E}\}) \triangleq \{\texttt{0}\}$$

Abstract Interpretation

$$\wp(\mathbb{P}) \quad \mapsto \quad \ldots \quad \mapsto \quad \wp(\mathbb{P})$$

$$\boxed{\texttt{calc : } \alpha(\uparrow\texttt{succ}(\gamma(\texttt{P}))) = \ldots \triangleq \uparrow\texttt{succ}^\sharp(\texttt{P})}$$

$$\uparrow\texttt{succ}^\sharp(\{\texttt{E}\}) \triangleq \{\texttt{0}\}$$

$$\wp(\mathbb{P}) \coloneqq (\mathbb{P} \to \texttt{prop}) = \text{“specification”}$$
$$\wp(\mathbb{P}) \coloneqq [\mathbb{P}] = \text{“constructed”}$$

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$\wp(\mathbb{P})$ ≔ $(\mathbb{P} \rightarrow$ prop) = “specification”
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$$\uparrow\texttt{succ}^\sharp(\{\texttt{E}\}) \triangleq \{\texttt{O}\}$$

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$$\boxed{\texttt{calc : } \alpha(\uparrow\texttt{succ}(\gamma(P))) = \ldots \triangleq \uparrow\texttt{succ}^\sharp(P)}$$

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$$\wp(\mathbb{P}) \coloneqq (\mathbb{P} \to \texttt{prop}) = \text{"specification"}$$

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no extraction

Abstract Interpretation

$$\wp(\mathbb{P}) \;\mapsto\; \dots \;\mapsto\; \wp(\mathbb{P})$$

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$$\gamma : \wp(\mathbb{P}) \to \wp(\mathbb{N})$$

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no extraction

Abstract Interpretation

General framework (calculation)

✓ framework

Mechanization Issues (no extraction)

✗ mechanize

Abstract Interpretation

γ-only (no calculation)

½ framework

Verification with extraction
Verasco [Jourdan et al POPL 2015]

✓ mechanize

Three Stories

Direct Verification	✗ `framework` ✓ `mechanize`
Abstract Interpretation	✓ `framework` ✗ `mechanize`
Constructive GCs	✓ `framework` ✓ `mechanize`

Constructive GCs

$$\wp(\mathbb{P}) \quad \mapsto \quad \dots \quad \mapsto \quad \wp(\mathbb{P})$$

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$$\uparrow\texttt{succ}^\sharp(\{E\}) \triangleq \{0\}$$

$$\wp(\mathbb{P}) \coloneqq (\mathbb{P} \to \texttt{prop}) \wedge \text{“has effects”}$$
$$\wp(\mathbb{P}) \coloneqq (\mathbb{P} \to \texttt{prop}) \wedge \text{“no effects”}$$

Constructive GCs

$\wp(\mathbb{P}) \quad \mapsto \quad \ldots \quad \mapsto \quad \wp(\mathbb{P})$

$$\boxed{\texttt{calc} : \alpha(\uparrow\mathrm{succ}(\gamma(P))) = \ldots \triangleq \uparrow\mathrm{succ}^\sharp(P)}$$

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$\wp(\mathbb{P}) \coloneqq (\mathbb{P} \to \texttt{prop}) \land$ "has effects"

$\wp(\mathbb{P}) \coloneqq (\mathbb{P} \to \texttt{prop}) \land$ "no effects"

$\wp(\mathbb{P}) \rightleftarrows \mathbb{P}$ (singleton in list monad)

Constructive GCs

$$\alpha^{M} : \mathbb{N} \rightarrow \wp(\mathbb{P})$$
$$\gamma^{M} : \mathbb{P} \rightarrow \wp(\mathbb{N})$$

vs

$$\alpha : \wp(\mathbb{N}) \rightarrow \wp(\mathbb{P})$$
$$\gamma : \wp(\mathbb{P}) \rightarrow \wp(\mathbb{N})$$

Constructive GCs

$\alpha^{M} : \mathbb{N} \rightarrow \wp(\mathbb{P})$

$\gamma^{M} : \mathbb{P} \rightarrow \wp(\mathbb{N})$

(+ monadic GC laws)

vs

$\alpha : \wp(\mathbb{N}) \rightarrow \wp(\mathbb{P})$

$\gamma : \wp(\mathbb{P}) \rightarrow \wp(\mathbb{N})$

(+ GC laws)

Constructive GCs

$$\alpha^{M} : \mathbb{N} \to \wp(\mathbb{P})$$
$$\gamma^{M} : \mathbb{P} \to \wp(\mathbb{N})$$

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$$\alpha : \wp(\mathbb{N}) \to \wp(\mathbb{P})$$
$$\gamma : \wp(\mathbb{P}) \to \wp(\mathbb{N})$$

(+ GC laws)

α^{M} “has no effects”

Constructive GCs

$$\alpha^{M} : \mathbb{N} \to \wp(\mathbb{P}) \qquad \alpha : \wp(\mathbb{N}) \to \wp(\mathbb{P})$$

$$\gamma^{M} : \mathbb{P} \to \wp(\mathbb{N}) \qquad \text{vs} \qquad \gamma : \wp(\mathbb{P}) \to \wp(\mathbb{N})$$

(+ monadic GC laws) (+ GC laws)

α^{M} “has no effects”

Constructive GCs

$\alpha^{M} : \mathbb{N} \to \wp(\mathbb{P})$
$\gamma^{M} : \mathbb{P} \to \wp(\mathbb{N})$

(+ monadic GC laws)

vs

$\alpha : \wp(\mathbb{N}) \to \wp(\mathbb{P})$
$\gamma : \wp(\mathbb{P}) \to \wp(\mathbb{N})$

(+ GC laws)

α^{M} "has no effects"

$$\boxed{\begin{array}{l} \eta : \mathbb{N} \to \mathbb{P} \\ \mu : \mathbb{P} \to \wp(\mathbb{N}) \end{array}}$$

Constructive GCs

$$\boxed{\begin{array}{lcl} \eta & : & \mathbb{N} \to \mathbb{P} \\ \mu & : & \mathbb{P} \to \wp(\mathbb{N}) \end{array}}$$

Constructive GCs

$$\boxed{\begin{array}{l} \eta \;:\; \mathbb{N} \rightarrow \mathbb{P} \\ \mu \;:\; \mathbb{P} \rightarrow \wp(\mathbb{N}) \end{array}}$$

$$\eta(n) \coloneqq \mathrm{parity}(n)$$
$$\mu(p) \coloneqq [\![p]\!]$$

Constructive GCs

$$\boxed{\begin{aligned} \eta &: \mathbb{N} \to \mathbb{P} \\ \mu &: \mathbb{P} \to \wp(\mathbb{N}) \end{aligned}}$$

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$\eta(n) \coloneqq \mathrm{parity}(n)$
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Constructive GCs

$$\boxed{\begin{array}{l} \eta : \mathbb{N} \to \mathbb{P} \\ \mu : \mathbb{P} \to \wp(\mathbb{N}) \end{array}}$$

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$\eta(n) \coloneqq \mathrm{parity}(n)$

$\mu(p) \coloneqq [\![p]\!]$

$\alpha(N) \coloneqq \{\eta(n) \mid n \in N\}$

$\gamma(P) \coloneqq \{n \mid p \in P \wedge n \in \mu(p)\}$

Constructive GCs

$\boxed{\texttt{calc : } (\alpha \circ {\uparrow}\mathrm{succ} \circ \gamma)(P) = {\uparrow}\mathrm{succ}^\sharp(P)}$

$\boxed{\texttt{calc : } (\lfloor\eta\rfloor \circledast \lfloor\mathrm{succ}\rfloor \circledast \mu)(p) = \lfloor\mathrm{succ}^\sharp\rfloor(p)}$

Constructive GCs

```
calc : (α∘↑succ∘γ)(P) = ↑succ♯(P)
```

```
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```

Constructive GCs

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calc : (α∘↑succ∘γ)(P) = ↑succ♯(P)
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calc : (⌊η⌋⊛⌊succ⌋⊛μ)(p) = ⌊succ♯⌋(p)
```

“powerset lifting = boilerplate”

Results

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- Sound, optimal and *computable* AIs by construction
- Metatheory and case studies all verified in Agda

Constructive GCs

$$\boxed{\begin{array}{l} \eta : \mathbb{N} \rightarrow \mathbb{P} \\ \mu : \mathbb{P} \rightarrow \wp(\mathbb{N}) \end{array}}$$

(+ monadic GC laws)

✓ `framework`
✓ `mechanize`

Draft: Constructive Galois Connections
http://arxiv.org/abs/1511.06965