Homework 1

Due: Friday, Jan 26, 11:59pm

Preface

Discussing high-level approaches to homework problems with your peers is encouraged. You must include at the top of your assignment a Collaboration Statement which declares any other people with whom you discussed homework problems. For example:

> *Collaboration Statement: I discussed problems 1 and 3 with Jamie Smith. I discussed problem 2 with one of the TAs. I discussed problem 4 with a personal tutor.*

If you did not discuss the assignment with anyone, you still must declare:

> *Collaboration Statement: I did not discuss homework problems with anyone.*

Copying answers or doing the work for another student is not allowed.

Assignment problems which refer to "Exercise X" or "Figure Y" are referring to those found in the Types and Programming Languages textbook.

Submitting

Prepare your assignment as either handwritten or using LaTeX. **I will not accept homework assignments written in Word, Google Docs, or using any other text processing software.** Handwritten assignments must be written neatly or they will receive a 0 grade. Submit the assignment either (1) via scanned pdf email to me: David.Darais@uvm.edu with "CS 225 HW1" in the subject line, or (2) placed under my office door (Votey 319) at any hour before the deadline.

Problem 1 (15 points)

Recall the definition for the divides relation:

$$\text{divides} \coloneqq \{\langle n, m\rangle \mid \exists o\ s.t.\ n \times o = m\}$$

Prove formally—and in as much detail as possible—that the divides relation is transitive, that is:

$$\text{forall } n\ m\ o, \text{if } n \text{ divides } m \text{ and } m \text{ divides } o,\ \text{then } n \text{ divides } o$$

You may assume basic algebraic arithmetic facts like $n + n = 2n$ and $2(n + n) = 2n + 2n$. Use the example proof of reflexivity given in class as a guide for the level of detail you should strive for.

Problem 2 (10 points)

Consider the set of boolean arithmetic terms $\mathcal{T}$ and metafunctions leaves (new) and depth (from Definition 3.3.2):

$$t \in \mathcal{T} ::= T \mid F \mid \texttt{if}\ t\ \texttt{then}\ t\ \texttt{else}\ t$$

$$\begin{aligned}
&\text{leaves}(T) \coloneqq 1\\
&\text{leaves}(F) \coloneqq 1\\
&\text{leaves}(\texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3) \coloneqq \text{leaves}(t_1) + \text{leaves}(t_2) + \text{leaves}(t_3)
\end{aligned}$$

$$\begin{aligned}
&\text{depth}(T) \coloneqq 1\\
&\text{depth}(F) \coloneqq 1\\
&\text{depth}(\texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3) \coloneqq \max(\text{depth}(t_1), \text{depth}(t_2), \text{depth}(t_3)) + 1
\end{aligned}$$

Define some term $t \in \mathcal{T}$ such that $\text{leaves}(t) = 7$ and $\text{depth}(t) = 3$.

Problem 3 (25 points)

Either prove by structural induction that leaves(t) always produces an odd number, or give a counter-example which shows leaves(t) can produce an even number.

Problem 4 (15 points)

Draw a derivation tree which justifies the following relationship:

$$\texttt{if}\ (\texttt{if}\ (\texttt{if}\ F\ \texttt{then}\ F\ \texttt{else}\ T)\ \texttt{then}\ T\ \texttt{else}\ F)\ \texttt{then}\ T\ \texttt{else}\ F$$
$$\longrightarrow$$
$$\texttt{if}\ (\texttt{if}\ T\ \texttt{then}\ T\ \texttt{else}\ F)\ \texttt{then}\ T\ \texttt{else}\ F$$

Problem 5 (30 points)

Consider the extended small-step semantics described in Exercise 3.5.16 which explicitly generates the value "`wrong`" in place of where the semantics from Figure 3-2 gets stuck.

1. Design a big-step semantics $t \Downarrow a$ (similar to 3.5.17) which is equivalent to this small-step semantics.
2. Prove that your new big-step semantics implies the small-step semantics which generates "`wrong`", that is, prove:

$$\textit{forall}\ t\ a,\ \ \textit{if}\ \ t \Downarrow a\ \ \textit{then}\ \ t \longrightarrow^{*} a$$

   This proof need not be as detailed as your answer to Problem 1, but still must be a convincing formal proof.

You should use the following syntactic categories for terms t, numeric values nv, values v, and answers a:

$$\begin{array}{rrl} t \in & \mathcal{T} ::= & T \mid F \mid \texttt{if}\ t\ \texttt{then}\ t\ \texttt{else}\ t \\ & \mid & 0 \mid \texttt{succ}\ t \mid \texttt{pred}\ t \mid \texttt{iszero}\ t \\ & \mid & \texttt{wrong} \end{array}$$

$$\begin{array}{rrl} nv \in & \mathcal{NV} ::= & 0 \mid \texttt{succ}\ nv \\ v \in & \mathcal{V} ::= & T \mid F \mid nv \\ a \in & \mathcal{A} ::= & v \mid \texttt{wrong} \end{array}$$

Extra Credit (15 points)

Prove the other direction of Problem 5, that is:

$$\textit{forall}\ t\ a,\ \ \textit{if}\ \ t \longrightarrow^{*} a\ \ \textit{then}\ \ t \Downarrow a$$

Problem 6 (5 points)

Approximately how many hours did you spend working on this assignment?